

CORONA PROBLEM WITH DATA IN IDEAL SPACES OF SEQUENCES

DMITRY V. RUTSKY

ABSTRACT. Let E be a Banach lattice on \mathbb{Z} with order continuous norm. We show that for any function $f = \{f_j\}_{j \in \mathbb{Z}}$ from the Hardy space $H_\infty(E)$ such that $\delta \leq \|f(z)\|_E \leq 1$ for all z from the unit disk \mathbb{D} there exists some solution $g = \{g_j\}_{j \in \mathbb{Z}} \in H_\infty(E')$, $\|g\|_{H_\infty(E')} \leq C_\delta$ of the Bézout equation $\sum_j f_j g_j = 1$, also known as the vector-valued corona problem with data in $H_\infty(E)$.

The (classical) Corona Problem (see, e. g., [8, Appendix 3]) has the following equivalent formulation: given a finite number of bounded analytic functions $f = \{f_j\}_{j=1}^N \subset H_\infty$ on the unit disk \mathbb{D} , is the condition $\inf_{z \in \mathbb{D}} \max_j |f_j(z)| > 0$ sufficient as well as necessary for the existence of some solutions $g = \{g_j\}_{j=1}^N \subset H_\infty$ of the Bézout equation $\sum_j f_j g_j = 1$? A positive answer to this problem was established for the first time by L. Carleson in [1], and later a relatively simple proof was found by T. Wolff (see, e. g., [8, Appendix 3]); we also mention another approach to the proof based on the theory of analytic multifunctions (see, e. g., [12]).

These important results set ground for many subsequent developments. One question to ask is what estimates are possible for the solutions g in terms of the estimates on f . In particular, estimates such as the $H_2(l^2)$ norm of g make it possible to extend these results to infinite sequences f . We formalize this problem somewhat vaguely for now; see Section 1 below for exact definitions. Notation $\langle f, g \rangle = \sum_j f_j g_j$ will be used with suitable sequences $f = \{f_j\}$ and $g = \{g_j\}$. Let E be a normed space of sequences, and let

$$E' = \{g \mid |\langle f, g \rangle| < \infty \text{ for all } f \in E\}$$

be the space of sequences dual to E with respect to the pairing $\langle \cdot, \cdot \rangle$. Suppose that $f \in H_\infty(E)$ satisfies $\delta \leq \|f(z)\|_E \leq 1$ for all $z \in \mathbb{D}$ with some $\delta > 0$. We are interested in the existence of a function $g \in H_\infty(E')$ satisfying $\langle f, g \rangle = 1$, and in the possible norm estimates of g in terms of δ . If this is the case for all such f then we say that E has the *corona property*.

T. Wolff's argument allowed M. Rosenblum, V. A. Tolokonnikov and A. Uchiyama to obtain (independently from one another) a positive answer

to this question in the Hilbert space case, showing that l^2 has the corona property. The corresponding estimates were later improved many times; apparently, the best one at the time of this writing can be found in [10]. A. Uchiyama also obtained in [11] a different estimate using a rather involved argument based on the original proof of L. Carleson, thus establishing that l^∞ also has the corona property. The intermediate spaces $E = l^p$, $2 < p < \infty$ between these two cases can be reduced to the case $p = 2$ (see [6, §3]), following Tolokonnikov's (unpublished) remark that Wolff's method can be directly extended to this case at least for even values of $p > 2$, but very little was known about the corona property for other spaces E .

Recently, in [5] S. V. Kislyakov extended the corona property to a large class of Banach ideal sequence spaces E . Specifically, in this result E is assumed to be q -concave with some $q < \infty$ and satisfy the Fatou property, and space $L_\infty(E)$ is assumed to be BMO-regular. These conditions are satisfied by all UMD lattices with the Fatou property, and in particular by spaces l^p , $1 \leq p < \infty$. A novel and somewhat counterintuitive idea leading to this result is that for suitable spaces E_0 and E_1 the corona property of E_0 implies the corona property of their pointwise product $E_0 E_1$. The proof uses the theory of interpolation for Hardy-type spaces to reduce the result to the well-known case $E = l^2$.

In the present work we show how the approach of [5] can be modified to obtain a complete answer: it turns out that *all* ideal sequence spaces with order continuous norm have the corona property. Note that this, in particular, includes all finite-dimensional ideal sequence spaces. These conditions are more general than the result [5]; see Proposition 9 at the end of Section 2. It remains unclear if the assumption of the order continuity of the norm can be weakened.

Compared to [5], our proof relies on somewhat less elementary means, namely we use a fixed point theorem and a selection theorem to reduce the problem to Uchiyama's difficult case $E = l^\infty$, but otherwise the reduction appears to be rather simple and straightforward. Moreover, for q -concave lattices E with $q < \infty$ the problem is still reduced in this manner to the relatively easy standard case $E = l^2$ using the same argument as in [5].

We also mention that very recently in [13] the method described in the present work was also applied to the problem of characterizing the ideals $I(f) = \{\langle f, g \rangle \mid g \in H_\infty(E')\}$. Certain classical results concerning the case $E = l^2$ were extended to the case $E = l^1$. The approach [5] based on

interpolation of Hardy-type spaces does not seem to lend itself to such an extension.

We briefly outline the implications for the estimates $C_{E,\delta}$ of the norms of the solutions g in terms of δ . Theorem 2 below can be stated quantitatively: $C_{E_0E_1,\delta} \leq C_{E_0,\frac{\delta}{2}}$ for $0 < \delta < 1$, provided that E_0E_1 is a Banach lattice with order continuous norm. Thus for any lattice E with order continuous norm we obtain an estimate

$$(1) \quad C_{E,\delta} \leq C_{l^\infty,\frac{\delta}{2}} \leq c_1 \delta^{-c_2}$$

with some constants $c_1, c_2 > 0$ independent of E . Furthermore, if a Banach lattice E is q -concave with some $1 < q < \infty$ then $E = l^q E_1$ with a Banach lattice E_1 (see the proof of Proposition 9 below), and we get an estimate $C_{E,\delta} \leq C_{l^q,\frac{\delta}{2}}$ that may be sharper than (1) for some values of q . Indeed, we also have an estimate $C_{l^p,\delta} \leq C_{l^2,\delta^{\frac{p}{2}}}$ for $p \geq 2$ (see, e. g., [6, §3]) and $C_{l^2,\delta} \leq \frac{1}{\delta} + c \frac{1}{\delta^2} \log \frac{1}{\delta}$ by [10] with an explicit constant $c \approx 8.4$. The latter estimate is known to be close to optimal in terms of the rate of growth as $\delta \rightarrow 0$. Thus, for q -concave lattices E with some $2 \leq q < \infty$ we also have an estimate $C_{E,\delta} \leq c \frac{1}{\delta^q} (\log \frac{1}{\delta})^{\frac{q}{2}}$ for small enough δ . Our knowledge about sharp estimates for the value of $C_{l^p,\delta}$ with $p \neq 2$ seems to be lacking.

1. STATEMENTS OF THE RESULTS

A quasi-normed *lattice* X of measurable functions on a measurable space Ω , also called an *ideal space*, is a quasi-normed space of measurable functions such that $f \in X$ and $|g| \leq |f|$ implies $g \in X$ and $\|g\|_X \leq \|f\|_X$. Ideal spaces of sequences E are lattices on $\Omega = \mathbb{Z}$. A lattice X is said to have *order continuous quasi-norm* if for any sequence $f_n \in X$ such that $\sup_n |f_n| \in X$ and $f_n \rightarrow 0$ almost everywhere one also has $\|f_n\|_X \rightarrow 0$. Lattices l^p have order continuous quasi-norm if and only if $p < \infty$. For a lattice X of measurable functions the *order dual* X' , also called the *associate space*, is the lattice of all measurable functions g such that the norm

$$\|g\|_{X'} = \sup_{f \in X, \|f\|_X \leq 1} \int |fg|$$

is finite. For example, the order dual of l^p is $l^{p'}$ for all $1 \leq p \leq \infty$. If X is a Banach lattice, the order dual is contained in the topological dual space X^* of all continuous linear functionals on X , and $X' = X^*$ if and only if X has order continuous norm. For more on lattices see, e. g., [4].

Definition 1. Suppose that E is a normed lattice on \mathbb{Z} . We say that E has the *corona property* with constant C_δ , $0 < \delta < 1$, if for any $f \in H_\infty(E)$ such

that¹ $\delta \leq \|f(z)\|_E \leq 1$ for all $z \in \mathbb{D}$ there exists some $g \in H_\infty(E')$ such that $\|g\|_{H_\infty(E')} \leq C_\delta$ and $\langle f(z), g(z) \rangle = 1$ for all $z \in \mathbb{D}$. Such a function f is called the data for the corona problem with lower bound δ , and such a function g is called the solution for the corona problem with data f .

For any two quasi-normed lattices E_0 and E_1 on the same measurable space the set of pointwise products

$$E_0 E_1 = \{h_0 h_1 \mid h_0 \in E_0, h_1 \in E_1\}$$

is a quasi-normed lattice with the quasi-norm defined by

$$\|h\|_{E_0 E_1} = \inf_{h=h_0 h_1} \|h_0\|_{E_0} \|h_1\|_{E_1}.$$

For example, the Hölder inequality shows that $l^p l^q = l^r$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. It is easy to see that $XL_\infty = X$ for any lattice X .

Theorem 2. *Suppose that E_0, E_1 are Banach lattices on \mathbb{Z} such that $E = E_0 E_1$ is also a Banach lattice having order continuous norm. If E_0 has the corona property with constants C_δ , $0 < \delta < 1$, then E also has it with constants $C_{\frac{\delta}{2}}$.*

The proof of Theorem 2 is given in Section 2 below. Since l^∞ has the corona property by [11], applying Theorem 2 with $E_0 = l^\infty$ and $E_1 = E$ yields the main result, stated as follows.

Theorem 3. *Every Banach lattice on \mathbb{Z} with order continuous norm has the corona property.*

2. PROOF OF THEOREM 2

We begin with some preparations. The following result shows that finite-dimensional approximation of the corona property is possible under the assumption that E has order continuous norm.

Proposition 4. *Suppose that E is a Banach lattice on \mathbb{Z} with order continuous norm such that for any $\varepsilon > 0$ and a finite $I \subset \mathbb{Z}$ the restriction of E onto I has the corona property with constants $(1 + \varepsilon)C_\delta$, $0 < \delta < 1$, where C_δ are independent of I and ε . Then E has the corona property with constants C_δ , $0 < \delta < 1$.*

¹ Replacing this condition with the nonstrict inequality $\delta < \|f(z)\|_E \leq 1$ would allow to simplify somewhat the arguments in Section 3 below; however, the closed form looks nicer.

The proof of Proposition 4 is given in Section 3 below.

A set-valued map $\Phi : X \rightarrow 2^Y$ between normed spaces is called *lower semicontinuous* if for any $x_n \in X$, $x_n \rightarrow x$ in X and $y \in \Phi(x)$ there exists a subsequence n' and some $y_{n'} \in \Phi(x_{n'})$ such that $y_{n'} \rightarrow y$ in Y . We need the following well-known result.

Michael Selection Theorem ([7]²). *Let Y be a Banach space, X a paracompact space and $\varphi : X \rightarrow 2^Y$ a lower semicontinuous multivalued map taking values that are nonempty, convex and closed. Then there exists a continuous selection $f : X \rightarrow Y$ of φ , i. e. $f(x) \in \varphi(x)$ for all $x \in X$.*

This allows us to conclude that the factorization corresponding to the product of finite-dimensional Banach lattices can be made continuous (in the more general infinite-dimensional cases this seems to be unclear).

Proposition 5. *Suppose that F_0 and F_1 are finite-dimensional Banach lattices of functions on the same measurable space. Then for every ε there exists a continuous map $\Delta : F_0 F_1 \setminus \{0\} \rightarrow F_1$ taking nonnegative values such that $\|\Delta f\|_{F_1} \leq 1$ and $\|f(\Delta f)^{-1}\|_{F_0} \leq (1 + \varepsilon)\|f\|_{F_0 F_1}$.*

Indeed, we first consider a set-valued map $\Delta_0 : F_0 F_1 \setminus \{0\} \rightarrow 2^{F_1}$ defined by

$$\Delta_0(f) = \{g \in F_1 \mid g > 0 \text{ everywhere,} \\ \|g\|_{F_1} < 1, \|fg^{-1}\|_{F_0} < (1 + \varepsilon)\|f\|_{F_0 F_1}\}$$

for $f \in F_0 F_1 \setminus \{0\}$. By the definition of the space $F_0 F_1$, the map Δ_0 takes nonempty values. It is easy to see that Δ_0 has open graph, and hence Δ_0 is a lower semicontinuous map. The graph of a map $\overline{\Delta}_0 : F_0 F_1 \setminus \{0\} \rightarrow 2^{F_1}$ defined by

$$\overline{\Delta}_0(f) = \{g \in F_1 \mid g \geq 0, \|g\|_{F_1} \leq 1, \|fg^{-1}\|_{F_0} \leq (1 + \varepsilon)\|f\|_{F_0 F_1}\}$$

(with the conventions $0 \cdot 0^{-1} = 0$ and $a \cdot 0^{-1} = \infty$ for $a \neq 0$) is easily seen to be the closure of the graph of the map of Δ_0 , therefore $\overline{\Delta}_0$ is also a lower semicontinuous map. The values of Δ_0 are convex and closed, so by the Michael selection theorem $\overline{\Delta}_0$ admits a continuous selection Δ , that is, $\Delta(f) \in \overline{\Delta}_0(f)$ for all $f \in F_0 F_1$. This selection satisfies the conclusion of Proposition 5.

Fan–Kakutani Fixed Point Theorem ([2]). *Suppose that K is a compact set in a locally convex linear topological space. Let Φ be a mapping from K to*

² For the sake of simplicity we omitted the converse part of this famous theorem.

the set of nonempty subsets of K that are convex and compact, and assume that the graph of Φ is closed. Then Φ has a fixed point, i. e. $x \in \Phi(x)$ for some $x \in K$.

A quasi-normed lattice X of measurable functions is said to have the *Fatou property* if for any $f_n, f \in X$ such that $\|f_n\|_X \leq 1$ and the sequence f_n converges to f almost everywhere it is also true that $f \in X$ and $\|f\|_X \leq 1$.

The following formula (also appearing in [5, Lemma 1]) seems to be rather well known; see, e. g., [9, Theorem 3.7].

Proposition 6. *Suppose that X and Y are Banach lattices of measurable functions on the same measurable space having the Fatou property such that XY is also a Banach lattice. Then $X' = (XY)'Y$.*

In order to achieve the best estimate possible with the method used without assuming that C_δ is continuous in δ , we take advantage of the fact that the decomposition in the definition of the pointwise product of Banach lattices can be made exact if both lattices satisfy the Fatou property.

Proposition 7. *Let X and Y be Banach lattices of measurable functions on the same measurable space having the Fatou property. Then for every function $f \in XY$ there exist some $g \in X$ and $h \in Y$ such that $f = gh$ and $\|g\|_X \|h\|_Y \leq \|f\|_{XY}$.*

This appears to be rather well known but hard to find in the literature, so we give a proof. We may assume that $\|f\|_{XY} = 1$. Let $\varepsilon_n \rightarrow 0$ be a decreasing sequence. Sets

$$F_n = \{g \mid g \geq 0, \text{ supp } g = \text{supp } f \text{ up to a set of measure } 0, \\ \|g\|_X \leq 1, \|fg^{-1}\|_Y \leq 1 + \varepsilon_n\} \subset X$$

are nonempty and form a nonincreasing sequence. It is easy to see that F_n are convex (one uses the convexity of the map $t \mapsto t^{-1}$, $t > 0$). By the Fatou property of X and Y sets F_n are closed with respect to the convergence in measure, and they are bounded in X . The intersection of such a sequence of sets is nonempty (see [4, Chapter 10, §5, Theorem 3]), so there exists some $g \in \bigcap_n F_n$, which together with $h = fg^{-1}$ yields the required decomposition.

Now we are ready to prove Theorem 2. Suppose that under its assumptions $E = E_0 E_1$, lattice E_0 has the corona property with constant C_δ for some $0 < \delta \leq 1$ and we are given some $f \in H_\infty(E)$ such that $\delta \leq \|f(z)\|_E \leq 1$ for all $z \in \mathbb{D}$; we need to find a suitable $g \in H_\infty(E')$ solving $\langle f, g \rangle = 1$.

Proposition 4 allows us to assume that the lattices have finite support $I \subset \mathbb{Z}$, and moreover, we may relax the claimed estimate for the norm of a solution to $(1 + \varepsilon)C_{\frac{\delta}{2}}$ for arbitrary $\varepsilon > 0$. It is easy to see that finite-dimensional lattices always have the Fatou property, so we may assume that both E_0 and E_1 , and thus both $L_\infty(E_0)$ and $L_\infty(E_1)$ have the Fatou property. By Proposition 7 there exist some $u \in L_\infty(E_0)$ and $v \in L_\infty(E_1)$ such that $|f| = uv$ and $\|u\|_{L_\infty(E_0)}\|v\|_{L_\infty(E_1)} \leq \|f\|_{L_\infty(E)} \leq 1$. We may further assume that $\|u\|_{L_\infty(E_0)} \leq 1$ and $\|v\|_{L_\infty(E_1)} \leq 1$.

Since $f = \{f_j\}_{j \in I}$ is analytic and bounded, if we restrict I so that $\mathbb{T} \times I$ becomes the support of f , we may assume that $\log |f_j| \in L_1$ for all $j \in I$. Boundedness of u and v further implies that $\log |v_j| \in L_1$ for all $j \in I$.

Let us fix some $\varepsilon > 0$ and a sequence $0 < r_j < 1$ such that $r_j \rightarrow 1$. We denote by P_r the operator of convolution with the Poisson kernel for radius $0 < r < 1$, that is, $P_r a(z) = a(rz)$ for any harmonic function a on \mathbb{D} and any $z \in \mathbb{D}$.

Let

$$(2) \quad B = \{\log w \mid w \in L_\infty(E_1), \|w\|_{L_\infty(E_1)} \leq 2, w \geq v\} \subset L_1.$$

This set is convex, which follows from the well-known logarithmic convexity of the norm of a Banach lattice. We endow B with the weak topology of L_1 . By the Fatou property of $L_\infty(E_1)$ it is easy to see that B is closed with respect to the convergence in measure, so B is also closed in L_1 and thus weakly closed. The Dunford–Pettis theorem easily shows that B is a compact set, since the functions from B are uniformly bounded from above and below by some summable functions.

For convenience, we denote by B_Z the closed unit ball of a Banach space Z . We endow $H_\infty(E'_0)$ with the topology of uniform convergence on compact sets in $\mathbb{D} \times I$, and define a (single-valued) map $\Phi_0^{(j)} : C_{\frac{\delta}{2}} B_{H_\infty(E'_0)} \rightarrow B$ by

$$\Phi_0^{(j)}(h) = \log(\Delta(|P_{r_j} h|) + v), \quad h \in C_{\frac{\delta}{2}} B_{H_\infty(E'_0)}$$

with a map Δ from Proposition 5 applied to $F_0 = E'$ and $F_1 = E_1$ (observe that by Proposition 6 we have $E'_0 = E'E_1$) and the chosen value of ε . It is easy to see that $\Phi_0^{(j)}$ is continuous.

We endow $H_\infty(E_1)$ with the topology of uniform convergence on compact sets in $\mathbb{D} \times I$ and define a (single-valued) map $\Phi_1 : B \rightarrow 2B_{H_\infty(E_1)}$ by

$$(3) \quad \Phi_1(\log w)(z, \omega) = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log w(e^{i\theta}, \omega) d\theta \right)$$

for all $\log w \in B$, $z \in \mathbb{D}$ and $\omega \in I$. This map is easily seen to be continuous and (since the integral under the exponent is the convolution with the Schwarz kernel) $|\Phi_1(\log w)| = w$ almost everywhere.

Observe that if $\psi = \Phi_1(\log w)$ for some $\log w \in B$ and $\varphi = \frac{f}{\psi}$ then $|\varphi| = \frac{|f|}{w} \leq \frac{|f|}{v} = u$ and we have $\varphi \in H_\infty(E_0)$ with $\|\varphi\|_{H_\infty(E_0)} \leq 1$. On the other hand,

$$\delta \leq \|f(z)\|_E = \|\varphi(z)\psi(z)\|_{E_0 E_1} \leq \|\varphi(z)\|_{E_0} \|\psi(z)\|_{E_1} \leq 2\|\varphi(z)\|_{E_0},$$

so $\frac{\delta}{2} \leq \|\varphi(z)\|_{E_0} \leq 1$ for all $z \in \mathbb{D}$. This means that φ belongs to the set

$$D = \left\{ \varphi \in H_\infty(E_0) \mid \frac{\delta}{2} \leq \|\varphi(z)\|_{E_0} \leq 1 \text{ for all } z \in \mathbb{D} \right\}$$

of corona data functions corresponding to the assumed corona property of E_0 . Thus we may define a (single-valued) map

$$\Phi_2 : \Phi_1(B) \rightarrow D$$

by $\Phi_2(\psi) = \frac{f}{\psi}$ for $\psi \in \Phi_1(B)$. We endow D with the topology of uniform convergence on compact sets in $\mathbb{D} \times I$. The continuity of Φ_2 is evident.

We define a set-valued map $\Phi_3 : D \rightarrow 2^{C_{\frac{\delta}{2}} B_{H_\infty(E'_0)}}$ by

$$\Phi_3(\varphi) = \left\{ h \in H_\infty(E'_0) \mid \langle \varphi, h \rangle = 1, \|h\|_{H_\infty(E'_0)} \leq C_{\frac{\delta}{2}} \right\}$$

for $\varphi \in D$. By the assumed corona property of E_0 map Φ_3 takes nonempty values. Since the condition $\langle \varphi, h \rangle = 1$ is equivalent to $\langle \varphi(z), h(z) \rangle = 1$ for all $z \in \mathbb{D}$, it is easy to see that the values of Φ_3 are convex and closed, and thus they are compact. Similarly, the closedness of the graph of Φ_3 is easily verified.

Now we define a set-valued map $\Phi^{(j)} : C_{\frac{\delta}{2}} B_{H_\infty(E'_0)} \rightarrow 2^{C_{\frac{\delta}{2}} B_{H_\infty(E'_0)}}$ by $\Phi^{(j)} = \Phi_3 \circ \Phi_2 \circ \Phi_1 \circ \Phi_0^{(j)}$. The graph of $\Phi^{(j)}$ is closed since all individual maps are continuous in the appropriate sense (specifically, as a composition of upper semicontinuous maps, but it is easy to establish the continuity in this case directly using compactness). The domain $C_{\frac{\delta}{2}} B_{H_\infty(E'_0)}$ with the introduced topology is a compact set in a locally convex linear topological space. Thus $\Phi^{(j)}$ satisfies the assumptions of the Fan–Kakutani fixed point theorem, which implies that the maps $\Phi^{(j)}$ admit some fixed points $h_j \in C_{\frac{\delta}{2}} B_{H_\infty(E'_0)}$, that is, $h_j \in \Phi^{(j)}(h_j)$ for all j . This means that with $\log w_j = \Phi_0^{(j)}(h_j)$, $\psi_j = \Phi_1(\log w_j)$ and $\varphi_j = \Phi_2(\psi_j)$ we have $h_j \in \Phi_3(\varphi_j)$. The first two conditions imply that $|\psi_j| = \Delta(|P_{r_j} h_j|) + v \geq \Delta(|P_{r_j} h_j|)$, so

$$\|(P_{r_j} h_j)(\psi_j)^{-1}\|_{L_\infty(E')} \leq \left\| \frac{|P_{r_j} h_j|}{\Delta(|P_{r_j} h_j|)} \right\|_{L_\infty(E')} \leq (1 + \varepsilon) C_{\frac{\delta}{2}}$$

by Proposition 5. Thus

$$(4) \quad \left\| \frac{h_j(r_j z)}{\psi_j(z)} \right\|_{E'} \leq (1 + \varepsilon) C_{\frac{\delta}{2}}$$

for all $z \in \mathbb{D}$, and condition $h_j \in \Phi_3(\varphi_j)$ implies that

$$(5) \quad 1 = \left\langle \frac{f(z)}{\psi_j(z)}, h_j(z) \right\rangle = \left\langle f(z), \frac{h_j(z)}{\psi_j(z)} \right\rangle$$

for all $z \in \mathbb{D}$. Since sequences ψ_j and h_j are uniformly bounded on compact sets in $\mathbb{D} \times I$, by passing to a subsequence we may assume that $\psi_j \rightarrow \psi$ with some $\psi \in \Phi_1(B)$ and $h_j \rightarrow h$ with some $h \in C_{\frac{\delta}{2}} B_{H_\infty(E'_0)}$ uniformly on compact sets in $\mathbb{D} \times I$. Thus we may pass to the limits in (4) and (5) to see that $\frac{h}{\psi}$ is a suitable solution for the corona problem with data f , which concludes the proof of Theorem 2.

We remark that this construction can be modified to use the Tychonoff fixed point theorem, which is the particular case of single-valued maps in the setting of the Fan–Kakutani theorem. It suffices to find a continuous selection for the slightly enlarged map Φ_3 , which is the purpose of the next result; the arbitrarily small increase in the estimate is inconsequential for the scheme of the proof.

Proposition 8. *Suppose that a finite-dimensional lattice E has the corona property with constant C_δ for some $0 < \delta < 1$. Let*

$$D_E = \{f \in H_\infty(E) \mid \delta \leq \|f(z)\|_E \leq 1 \text{ for all } z \in \mathbb{D}\}.$$

Then for any $\varepsilon > 0$ there exists a continuous map

$$K : D_E \rightarrow (1 + \varepsilon) C_\delta B_{H_\infty(E')}$$

such that $\langle f, K(f) \rangle = 1$ for any $f \in D_E$.

Indeed, let $0 < \alpha < 1$. We define a set-valued map

$$K_0 : D_E \rightarrow (1 + \alpha) C_\delta B_{H_\infty(E')}$$

by

$$K_0(f) = \{g \in H_\infty(E') \mid \|\langle f, g \rangle - 1\|_{H_\infty} < \alpha, \|g\|_{H_\infty(E')} < (1 + \alpha) C_\delta\}$$

for $f \in D_E$. By the corona property assumption on E map K_0 takes nonempty values. It is easy to see that K_0 has open graph and thus K_0 is a lower semicontinuous map. The graph of a map $\overline{K}_0 : D_E \rightarrow (1 + \alpha) C_\delta B_{H_\infty(E')}$ defined by

$$\overline{K}_0(f) = \{g \in H_\infty(E') \mid \|\langle f, g \rangle - 1\|_{H_\infty} \leq \alpha, \|g\|_{H_\infty(E')} \leq (1 + \alpha) C_\delta\}$$

is easily seen to be the closure of the graph of the map of K_0 , and hence $\overline{K_0}$ is also a lower semicontinuous map. The values of K_0 are convex and closed. By the Michael selection theorem there exists a continuous selection K_1 of the map $\overline{K_0}$, that is, $K_1(f) \in \overline{K_0}(f)$ for all $f \in D_E$. Now observe that $|\langle f(z), K_1(f)(z) \rangle - 1| \leq \alpha$ implies $|\langle f(z), K_1(f)(z) \rangle| \geq 1 - \alpha$ for all $z \in \mathbb{D}$ and $f \in D_E$, so we may set $K(f) = \frac{K_1(f)}{\langle f, K_1(f) \rangle}$ and have $\langle f, K(f) \rangle = 1$ with $\|K(f)\|_{H_\infty(E')} \leq \frac{1+\alpha}{1-\alpha} C_\delta$. Choosing α small enough yields the claimed range of K .

Finally, we mention that Theorem 3 includes the result [5, Corollary 2]. This is implied by the following known observation; we give a proof for convenience.

Proposition 9. *Suppose that X is a Banach lattice of measurable functions having the Fatou property and X is q -concave with some $1 < q < \infty$. Then X has order continuous norm.*

Lattice Z^δ is defined by the norm $\|f\|_{Z^\delta} = \left\| |f|^{\frac{1}{\delta}} \right\|_Z^\delta$ for a quasi-normed lattice Z of measurable functions and $\delta > 0$. Lattice X' is q' -convex, and hence $Y = (X')^{q'}$ is a Banach lattice with the Fatou property. Then $X' = Y^{\frac{1}{q'}}$. The Fatou property is equivalent to the order reflexivity $X = X''$, and using the well-known formula for the duals of the Calderón-Lozanovsky products (see, e. g., [9, Theorem 2.10]) we may write

$$X = (X')' = \left(Y^{\frac{1}{q'}} L_\infty^{\frac{1}{q'}} \right)' = Y'^{\frac{1}{q'}} L_1^{\frac{1}{q'}} = Y'^{\frac{1}{q'}} L_q.$$

Since lattice L_q has order continuous norm, it suffices to establish the following.

Proposition 10. *Suppose that X and Y are quasi-normed lattices of measurable functions and Y has order continuous quasi-norm. Then XY also has order continuous quasi-norm.*

Let $f_n \in XY$ be a sequence with $f = \sup_n |f_n| \in XY$ such that $f_n \rightarrow 0$ almost everywhere. Then $f = gh$ with some $g \in X$ and $h \in Y$. We may assume that $g, h \geq 0$. Sequence $h_n = \frac{f_n}{g}$ also converges to 0 almost everywhere, and $|h_n| \leq \frac{f}{g} = h$, so $\sup_n |h_n| \in Y$. By the order continuity of the quasi-norm of Y we have $\|h_n\|_Y \rightarrow 0$, hence $\|f_n\|_{XY} \leq \|g\|_X \|h_n\|_Y \rightarrow 0$.

3. PROOF OF PROPOSITION 4

First, observe that if a lattice E on \mathbb{Z} has order continuous norm then $L_1(E)$ also has order continuous norm, we have $L_\infty(E') = [L_1(E)]' = [L_1(E)]^*$, and $H_\infty(E')$ is easily seen to be w^* -closed in $L_\infty(E')$ (see, e. g., [6,

§1.2.1]). The w^* -convergence of a sequence $h_k \in H_\infty(E')$ to some h implies that $h_k(z) \rightarrow h(z)$ in the $*$ -weak topology of $E' = E^*$ for all $z \in \mathbb{D}$.

Now suppose that under the assumptions of Proposition 4 $0 < \delta < 1$ and $f \in H_\infty(E)$ satisfies $\delta \leq \|f(z)\|_E \leq 1$ for all $z \in \mathbb{D}$. Let $I_k \subset \mathbb{Z}$ be a nondecreasing sequence such that $\bigcup_k I_k = \mathbb{Z}$, and fix a sequence $\varepsilon_j > 0$, $\varepsilon_j \rightarrow 0$. We consider the natural approximations $f_{A,r,k}(z) = Af(rz)\chi_{I_k}$, $z \in \mathbb{D}$, for $0 < r \leq 1$ and $A \geq 1$. If there exists some sequence of parameters $A_j \rightarrow 1$, $r_j \rightarrow 1$ and $k_j \rightarrow \infty$ such that $f_j = f_{A_j,r_j,k_j}$ is a data for the corona problem with lower bound δ then by the assumptions there exist some g_j such that $\langle f_j, g_j \rangle = 1$ and $\|g_j\|_{H_\infty(E')} \leq (1 + \varepsilon_j)C_\delta$. By passing to a subsequence we may assume that A_j is nonincreasing, r_j and k_j are nondecreasing, and $g_j \rightarrow g$ in the $*$ -weak topology of $L_\infty(E')$ for some $g \in H_\infty(E')$. Observe that

$$(6) \quad 1 = \langle f_j(z), g_j(z) \rangle = \langle f(z), g_j(z) \rangle + \langle f(r_j z) - f(z), g_j(z) \rangle + \langle f_j(z) - f(r_j z), g_j(z) \rangle$$

for all $z \in \mathbb{D}$. The first term in (6) converges to $\langle f(z), g(z) \rangle$. Since the E -valued analytic function f is strongly continuous at every $z \in \mathbb{D}$, the second term in (6) converges to 0. By the assumptions

$$(7) \quad |f_j(z) - f(r_j z)| = |A_j f(r_j z)\chi_{I_{k_j}} - f(r_j z)| \leq \chi_{I_{k_j}} |A_j f(r_j z)\chi_{I_{k_j}} - f(r_j z)| + \chi_{\mathbb{Z} \setminus I_{k_j}} |A_j f(r_j z)\chi_{I_{k_j}} - f(r_j z)| = \chi_{I_{k_j}} |A_j f(r_j z) - f(r_j z)| + \chi_{\mathbb{Z} \setminus I_{k_j}} |f(r_j z)| \leq (A_j - 1)|f(r_j z)| + \chi_{\mathbb{Z} \setminus I_{k_j}} |f(z)| + |f(r_j z) - f(z)|.$$

The norm in E of the first term in (7) is estimated by $(A_j - 1)$, and thus it converges to 0. The second term in (7) converges to 0 in E by the assumption that E has order continuous norm. The third term in (7) converges to 0 in E by the strong continuity of f in \mathbb{D} . It follows that the third term in (6) is dominated by $\|f_j(z) - f(r_j z)\|_E \|g_j(z)\|_{E'} \leq \|f_j(z) - f(r_j z)\|_E (1 + \varepsilon_j)C_\delta$, and so it also converges to 0. Therefore passing to the limit in (6) yields $\langle f(z), g(z) \rangle = 1$ for all $z \in \mathbb{D}$. We also have $\|g\|_{H_\infty(E')} \leq \limsup_j \|g_j\|_{H_\infty(E')} \leq C_\delta$, so g is a solution for the corona problem with data f having the claimed constant C_δ .

Thus it suffices to find a suitable sequence of parameters. We consider two cases. In the first case $\|f(z)\|_E = 1$ for all $z \in \mathbb{D}$. We take $A_j = 1$ and any increasing sequence $r_j \rightarrow 1$. By the order continuity of norm we have $\|f(z)\chi_{I_k}\|_E \rightarrow \|f(z)\|_E = 1$ for every $z \in \mathbb{D}$, so by the compactness of

closed sets in \mathbb{D} and the assumption that $\delta < 1$ we have $\|f(r_j z)\chi_{I_k}\|_E \geq \delta$ for large enough k_j . Thus f_j is a suitable corona data in this case.

In the second case $\|f(z_0)\|_E < 1$ for some $z_0 \in \mathbb{D}$. With the help of an automorphism we may assume for convenience that $z_0 = 0$. We also fix any increasing sequence $r_j \rightarrow 1$. A simple consequence of the Schwarz lemma (see, e. g., [3, Chapter 1, Corollary 1.3]) shows that

$$(8) \quad |\langle f(z), e' \rangle| \leq \frac{|\langle f(0), e' \rangle| + |z|}{1 + |\langle f(0), e' \rangle| |z|}$$

for all $z \in \mathbb{D}$ and $e' \in E' = E^*$ with $\|e'\|_{E'} \leq 1$. Since function $(x, y) \mapsto \frac{x+y}{1+xy}$ is increasing in both $x \in [0, 1]$ and $y \in [0, 1]$, taking the supremum in (8) over all such e' yields $\|f(z)\|_E \leq \frac{\|f(0)\|_E + |z|}{1 + \|f(0)\|_E |z|}$, thus $\alpha_j = \sup_{z \in \mathbb{D}} \|f(r_j z)\|_E < 1$. Setting $A_j = \frac{1}{\alpha_j}$ yields $\|A_j f(r_j z)\|_E \leq 1$ for all $z \in \mathbb{D}$. Again, since E has order continuous norm we have $\|A_j f(z)\chi_{I_k}\|_E \rightarrow \|A_j f(z)\|_E \geq A_j \delta > \delta$ for every $z \in \mathbb{D}$, and we also have $\left\|A_j f(r_j z)\chi_{I_{k_j}}\right\|_E \geq \delta$ for large enough k_j , so f_j is a suitable corona data in this case as well. The proof of Proposition 4 is complete.

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REFERENCES

- [1] L. CARLESON. Interpolations by bounded analytic functions and the corona problem. *Ann. Math.* 76, 2 (1962), 547–559.
- [2] K. FAN. Fixed-point and minimax theorems in locally convex topological linear spaces. *Proc. Nat. Acad. Sci. U.S.A.* 38 (1952), 121–126.
- [3] J. B. GARNETT. *Bounded analytic functions*. Academic Press New York, 1981.
- [4] L. V. KANTOROVICH, G. P. AKILOV. *Functional Analysis*, 2nd ed. Nauka, Moscow, 1977. In Russian; English transl.: Pergamon Press, Oxford–New York, 1982.
- [5] S. V. KISLYAKOV. Corona theorem and interpolation. *Algebra i Analiz* 27, 5 (2015), 69–80. In Russian; English transl.: St. Petersburg Math. J., 27 (2016), 757–764.
- [6] S. V. KISLYAKOV, D. V. RUTSKY. Some remarks to the corona theorem. *Algebra i Analiz* 24, 2 (2012), 171–191. In Russian; English transl.: St. Petersburg Math. J., 24 (2013), 313–326.
- [7] E. MICHAEL. Continuous selections. I. *Ann. Math. Second Series* 63, 2 (1956), 361–382.
- [8] N. K. NIKOL'SKIĬ. *Treatise on the Shift Operator*. Springer-Verlag, Berlin, Heidelberg, New York, 1986.
- [9] A. R. SCHEP. Products and factors of Banach function spaces. *Positivity* 14 (2010), 301–319.
- [10] S. TREIL, B. D. WICK. The matrix-valued H^p corona problem in the disk and polydisk. *J. Funct. Anal.* 226, 1 (2005), 138–172.
- [11] A. UCHIYAMA. Corona theorems for countably many functions and estimates for their solutions. Preprint, UCLA, 1980.

- [12] Z. SŁODKOWSKI. An analytic set-valued selection and its applications to the corona theorem, to polynomial hulls and joint spectra. *Trans. Am. Math. Soc.* 294, 1 (1986), 367–377.
- [13] I. K. ZLOTNIKOV. Estimates in problem of ideals in the algebra H^∞ . *Zap. Nauchn. Sem. POMI* 447 (2016), 66–74. In Russian.

STEKLOV MATHEMATICAL INSTITUTE, ST. PETERSBURG BRANCH, FONTANKA 27,
191023 ST. PETERSBURG, RUSSIA
E-mail address: `rutsky@pdmi.ras.ru`